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Growth and Roughness of the Interface for Ballistic Deposition

Mathew D. Penrose

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Abstract In ballistic deposition (BD), $(d+1)$-dimensional particles fall sequentially at random towards an initially flat, large but bounded d-dimensional surface, and each particle sticks to the first point of contact. For both lattice and continuum BD, a law of large numbers in the thermodynamic limit establishes convergence of the mean height and surface width (sample standard deviation of the height) of the interface to constants $h(t)$ and $w(t)$, respectively, depending on time t. We show that $h(t)$ is asymptotically linear in t, while $(w(t))^2$ grows at least logarithmically in t when $d = 1$. We use duality results showing that $w(t)$ can be interpreted as the standard deviation of the height for deposition onto a surface growing from a single point.

Keywords Stochastic growth model · First-passage percolation

1 Introduction

Scientific interest in growth processes associated with the deposition of particles on surfaces is considerable; see Barabási and Stanley [3], Cumberland and Crawford [5], Vicsek [22]. One family of deposition models involves $(d+1)$-dimensional particles which rain down sequentially at random onto a d-dimensional substrate (surface); when a particle arrives on the existing agglomeration of deposited particles, it sticks to the first particle it contacts, which may result in lateral growth and 'overhangs' (if it does not contact any previous particle, it sticks to the substrate). This is known as *ballistic deposition* (BD), and one reason for studying it is as a more tractable modification of *diffusion limited aggregation* (see Atar *et al.* [1]).

The physical sciences literature concerned with ballistic deposition in both lattice and continuum settings is extensive (see [3] for an overview). As well as numerous simulation studies dating back to Vold [23], this literature contains analysis by means of scaling theory (Family [6], Family and Vicsek [7], Kardar *et al.* [10]). However, these arguments have not

M.D. Penrose (✉)
Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, England, UK
e-mail: masmdp@maths.bath.ac.uk

been made rigorous; see, e.g., p. 56 of [3]. The rigorous mathematical literature is much less extensive, but see [1, 17, 18, 20]. The present article builds on the approach of Penrose and Yukich [17, 18].

In all versions of BD considered here, the substrate is the set $\mathbb{R}^d \times \{0\}$, identified with $\mathbb{R}^d$, or is some sub-region thereof (denoted Q). Thus the substrate is assumed initially flat. All particles are $(d+1)$-dimensional solids (typically balls or cubes). Particles arrive sequentially at random positions in $\mathbb{R}^d$. When a particle arrives at a position $x \in \mathbb{R}^d$, its centroid (or some other specified point in the particle identifying its location) slides instantaneously down the ray $\{x\} \times [0, \infty)$, starting from infinity, until the particle hits a position adjacent to either the substrate or a previously deposited particle, at which point its motion stops and it is permanently fixed. The difference between lattice and continuum models is that in the lattice model the positions at which particles arrive are restricted to be in the integer lattice $\mathbb{Z}^d$ (embedded in $\mathbb{R}^d$).

One is interested in the height and width (roughness) of the interface consisting of 'exposed' particles that are 'visible from above'. Loosely speaking, the height and width are interpreted here as the sample mean and standard deviation of the heights of exposed particles. Let $W_{t,n}$ denote the width for deposition onto a d-dimensional box of volume n, running for "time" (i.e., average number of particles deposited per unit volume of the box) t. Scaling theory [6, 7, 10] predicts, and subsequent experimental and theoretical studies seem to confirm (for an overview, see [3, 22]) that there exist a roughness exponent α and a growth exponent β, such that the surface width is governed by the dynamic scaling relation

$$W_{t,n} \approx n^\alpha f(t/n^{\alpha/\beta}) \tag{1.1}$$

where the scaling function f satisfies $f(x) \propto x^\beta$, for $x \ll 1$, and $f(x) \approx C$ for $x \gg 1$. If the scaling theory is correct, it implies that

$$W_{t,n} \propto t^\beta, \quad t \ll n; \qquad W_{t,n} \propto n^\alpha, \quad t \gg n.$$

Physicists believe (see e.g. p. 56 of [3]), that dimension $d = 1$ one has the 'exact' values $\alpha = 1/2$ and $\beta = 1/3$ (higher dimensional values of α and β are not known). Validating the above theory rigorously is a challenge for mathematicians; in the present work we take some steps in the direction of rigorously analyzing the regime with $t \ll n$.

In this paper we consider both lattice and continuum BD models. As in [18] we first take a thermodynamic limit (deposition onto an infinite surface). In this limit we obtain expressions for the limiting height and width, as a function of 'time' (the mean number of particles deposited per unit area). The next step is to examine the growth of these functions with time, and in doing so we go beyond the continuum analysis in [18]. We show that the limiting height grows asymptotically linearly with time and that in one dimension, the squared limiting width grows at least logarithmically.

The limits are taken in the opposite order in [1], while the time-parameter and the dimensions of the surface are simultaneously re-scaled in Seppäläinen [20]. Gravner *et al.* [8, 9] provide detailed results on a one dimensional discrete-time growth model sharing some features with BD, which are consistent with the belief that $\beta = 1/3$. Also, the well-known *first passage percolation* model shares features with BD, both in its description and in the questions one considers. Some of the related literature on first passage percolation is discussed at the end of Sect. 2.

2 Lattice Ballistic Deposition

We first consider a class of lattice ballistic deposition models, in which all particles are assumed identical. Let $\mathbf{0}$ denote the origin in $\mathbb{Z}^d$. Specify a *displacement function* $D : \mathbb{Z}^d \mapsto [-\infty, \infty)$ with the properties that (i) $D(\mathbf{0}) = 1$, and (ii) the set $\mathcal{N} := \{x \in \mathbb{Z}^d : D(x) \neq -\infty\}$ is finite but has at least two elements (one of which is the origin). For $x \in \mathbb{Z}^d$ let $\mathcal{N}_x := \{x + y : y \in \mathcal{N}\}$ and let $\mathcal{N}_x^* := \{x - y : y \in \mathcal{N}\}$. The set $\mathcal{N}$ is a 'neighbourhood' of the origin and $\mathcal{N}_x$ is the corresponding neighborhood of x. The idea of a displacement function is that if a particle arrives at $y \in \mathcal{N}_x$, then it cannot slide down the ray $y \times [0, \infty)$ below the position at height $D(y - x)$.

The substrate is represented by a finite subset Q of $\mathbb{Z}^d$ with $|Q|$ elements. At each site $x \in Q$, particles arrive at times forming a homogeneous Poisson process of unit rate, independently of other sites. We consider two alternative measures of the height of the interface at site x, the *last-arrival* height $\xi_{t,Q}(x)$ and the *next-arrival* height $\eta_{t,Q}(x)$. The latter is defined in terms of the former by

$$\eta_{t,Q}(x) := \max\{\xi_{t,Q}(y) + D(x - y) : y \in \mathbb{Z}^d\}, \tag{2.1}$$

where we set $-\infty + x := -\infty$ for $x \in \mathbb{R}$, so in fact $\eta_{t,Q}(x) = \max\{\xi_{t,Q}(y) + D(x - y) : y \in \mathcal{N}_x^*\}$. See Fig. 1 for an illustration of these heights.

The evolution of $\xi_{t,Q}(\cdot)$ proceeds as follows. Assume $\xi_{0,Q}(z) = 0$ for all $z \in \mathbb{Z}^d$, and as a function of t, assume $\xi_{t,Q}(z)$ is right-continuous and piecewise constant with jumps only at the arrival times of the Poisson process of arrivals at site z. If a particle arrives at site z at time t, then at time t the (last-arrival) height at site z is updated to the next-arrival height just before time t, i.e. we set

$$\xi_{t,Q}(x) = \eta_{t-,Q}(x) := \lim_{s \uparrow t} \eta_{s,Q}(x),$$

while the last-arrival heights at other sites remains unchanged, i.e., $\xi_{t,Q}(y) = \xi_{t-,Q}(y)$ for $y \neq z$.

Special cases include the so-called nearest-neighbor (NN) and next-nearest neighbor (NNN) models [3]. In the NN model, one takes $\mathcal{N} = \{z \in \mathbb{Z}^d : \|z\|_1 \leq 1\}$ (i.e., the origin

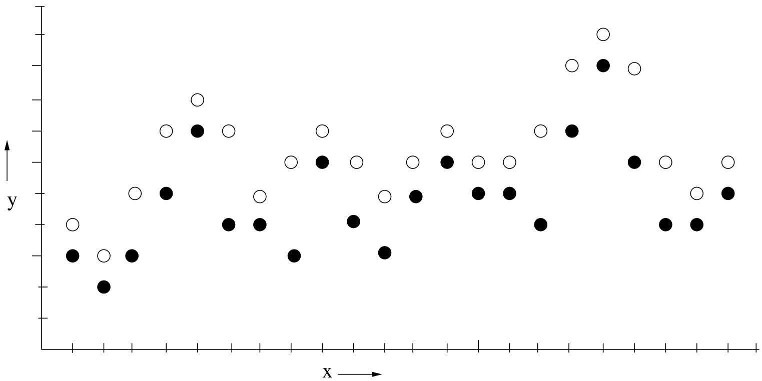

Fig. 1 The *dark dots* represent the graph of the last-arrival height $y = \xi_{t,Q}(x)$, and the *light dots* represent the next-arrival height $y = \eta_{t,Q}(x)$, in a realization of the NN model with $d = 1$ and $Q = [1, 22] \cap \mathbb{Z}$

together with its lattice neighbors), and the displacement function D is given by $D(x)=0$ for $x \in \mathcal{N} \setminus \mathbf{0}$; this is the version of ballistic deposition considered in [20], and it is this model which is illustrated in Fig. 1. In the NNN model, one takes $\mathcal{N}=\{z \in \mathbb{Z}^d : \|z\|_\infty \leq 1\}$ (i.e., diagonal neighbors are included) and takes $D(x)=1$ for all $x \in \mathcal{N}$; this is the version considered in [1].

Define the *mean height functional* $\overline{\xi}_{t,Q}$ and *width functional* $W_{t,Q}$ to be the sample mean and sample variance, respectively, of the heights at time t, i.e., set

$$\overline{\xi}_{t,Q} := |Q|^{-1} \sum_{z \in Q} \xi_{t,Q}(z); \qquad W_{t,Q} := \sqrt{|Q|^{-1} \sum_{x \in Q} (\xi_{t,Q}(x) - \overline{\xi}_{t,Q})^2}. \tag{2.2}$$

The mean height $\overline{\xi}_{t,Q}$ is a measure of the average amount of empty space under the surface, while the surface width functional $W_{t,Q}$ is a measure of the roughness of the interface.

We consider a particular limiting regime. First we take Q to be large with t fixed (deposition for a finite time onto a very large surface), and then we would like to take the large-time limit.

The large-Q limit of $\overline{\xi}_{t,Q}$ and $W_{t,Q}$ is best described in terms of deposition onto an 'infinite substrate' represented by the whole of $\mathbb{Z}^d$. Let $\xi_t(z)$ be the height above site z of this infinite interface at time $t \geq 0$. Assume again that $\xi_t(z)=0$ for all $z \in \mathbb{Z}^d$, but now assume particles arrive as independent Poisson processes at *all* sites in $\mathbb{Z}^d$. With this as the only difference from the description of $\xi_{t,Q}(\cdot)$, let the updating rules for $\xi_t(\cdot)$ be just the same as before. Also, define the next-arrival height $\eta_t(x)$ in an analogous manner to (2.1). For the infinite interface process we need to check that no 'explosions' occur; our first result does this and more.

Proposition 2.1 *For all $t \in (0,\infty)$, the values of $\xi_t(\mathbf{0})$ and $\eta_t(\mathbf{0})$ are almost surely finite, and for all $k \in \mathbb{N}$, it is the case that $E[(\xi_t(\mathbf{0}))^k] = O(t^k)$ and $E[(\eta_t(\mathbf{0}))^k] = O(t^k)$ as $t \to \infty$.*

We can interpret $\xi_t(\mathbf{0})$ as the height of a 'typical' point in the infinite interface. The next result is a thermodynamic limit and shows that in the large n limit, the height functional and squared width functional converge to the mean and variance, respectively, of the 'typical' height $\xi_t(\mathbf{0})$ (analogously to results in [18] for continuum BD).

Let $(Q_n, n \geq 1)$ be a sequence of finite subsets of $\mathbb{Z}^d$. Let ∂Q_n denote the set of boundary sites in Q_n. Assume that

$$\mathbf{0} \in Q_n \quad \text{for all } n; \tag{2.3}$$

$$\liminf(Q_n) = \mathbb{Z}^d, \quad i.e. \quad \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty Q_m = \mathbb{Z}^d; \tag{2.4}$$

$$|\partial Q_n|/|Q_n| \to 0 \quad \text{as } n \to \infty. \tag{2.5}$$

For example, Q_n could be a lattice box of side n centered at the origin.

Proposition 2.2 *For all $p \in [1,\infty)$ and all $t \in (0,\infty)$,*

$$\overline{\xi}_{t,Q_n} \xrightarrow{L^p} E[\xi_t(\mathbf{0})] := h(t) \quad \text{as } n \to \infty; \tag{2.6}$$

$$W^2_{t,Q_n} \xrightarrow{L^p} \mathrm{Var}[\xi_t(\mathbf{0})] := w^2(t) \quad \text{as } n \to \infty. \tag{2.7}$$

It is possible to give central limit theorems associated with the above laws of large numbers. For continuum BD, such results have been given in [18]. Similar arguments apply in the lattice case under consideration here, where one can use the general lattice central limit theorem of Penrose ([16], Theorem 3.1).

It is of great interest to estimate the limiting constants $h(t)$ and $w(t)$ in Proposition 2.2, and especially, to understand the growth of $h(t)$ and $w(t)$ as t becomes large. Our main results are concerned with this. Heuristically, one expects $h(t)$ to grow linearly in t since the expected height should vary directly with the deposition intensity t. The next result demonstrates this, and more.

Theorem 2.3 *There is a constant* $\rho_1 \in (0, \infty)$ *such that* $t^{-1}h(t) \to \rho_1$ *as* $t \to \infty$. *Moreover, for any* $p \in [1, \infty)$ *we have the* L^p *convergence*

$$t^{-1}\xi_t(\mathbf{0}) \to \rho_1 \quad \text{as } t \to \infty. \tag{2.8}$$

In the special case of the NN model, it can be deduced from Theorem 1 of Seppäläinen [20] that (2.8) holds with almost sure convergence. Our approach is somewhat different from that of [20], and is needed for subsequent results.

As for the width, the scaling theory mentioned in Sect. 1 predicts that $w^2(t) = \Theta(t^{2\beta})$. Even without scaling theory, one expects at least that $w^2(t) = O(t)$ on the basis of simulations of Zabolitzky and Stauffer ([24], p. 1529), and also on the following heuristic grounds. If $\mathcal{N} = \{\mathbf{0}\}$ then the heights $\xi_{t,n}(x), x \in Q_n$ are independent Poisson variables so that $w^2(t) = t$. If $\mathcal{N} \neq \{\mathbf{0}\}$ so as to give non-trivial interactions, these interactions should have a 'smoothing' effect so that $w^2(t)$ should not be any bigger than in the case $\mathcal{N} = \{\mathbf{0}\}$.

Rigorous analysis of the large-t behavior of $w(t)$ appears to be difficult: the following result makes a start.

Theorem 2.4 *For any* d, *it is the case that*

$$\liminf_{t\to\infty} w^2(t) > 0, \tag{2.9}$$

and if $d = 1$ *and* $\mathcal{N}$ *is a lattice interval, then*

$$\liminf_{t\to\infty}(w^2(t)/\log t) > 0. \tag{2.10}$$

The proof of Theorem 2.4 uses a *duality* relation between the ballistic deposition process and a *dual BD* process, denoted $(\hat{\xi}_t(x), x \in \mathbb{Z}^d)_{t\geq 0}$ with corresponding next-arrival process $(\hat{\eta}_t(x), x \in \mathbb{Z}^d)_{t\geq 0}$, defined in an identical manner to the original BD process $\xi_t(x)$ and next-arrival process $\eta_t(x)$, except that now one uses the dual displacement function $\hat{D}$ given by

$$\hat{D}(x) = D(-x), \quad x \in \mathbb{Z}^d, \tag{2.11}$$

and takes as initial configuration a single particle at height 0 at the origin, i.e., one takes

$$\hat{\xi}_0(x) = \begin{cases} 0 & \text{if } x = \mathbf{0}, \\ -\infty & \text{otherwise.} \end{cases}$$

We also define a further process $(\tilde{\eta}_t(x), x \in \mathbb{Z}^d)_{t\geq 0}$, by

$$\tilde{\eta}_t(x) = \hat{\eta}_{(t-T)^+}(x)$$

where T is exponentially distributed with mean 1, independent of the process $(\hat{\eta}_s(x), x \in \mathbb{Z}^d)_{s\geq 0}$. In other words, the process $(\tilde{\eta}_t(\cdot), t \geq 0)$ is obtained by waiting an exponentially distributed amount of time before 'kicking off' the dual next-arrival process $\hat{\eta}$. We shall refer to $\tilde{\eta}$ as the *delayed dual BD process*.

Theorem 2.5 *Let $t > 0$. Then the distribution of $\eta_t(\mathbf{0})$ is the same as that of the maximum depth $\sup_{z\in\mathbb{Z}^d} \hat{\eta}_t(z)$ in the dual BD process. The distribution of $\xi_t(\mathbf{0})$ is the same as that of the maximum depth $\sup_{z\in\mathbb{Z}^d} \tilde{\eta}_t(z)$ in the delayed dual BD process.*

Theorem 2.5 shows that $w^2(t) = \mathrm{Var}(\sup_{z\in\mathbb{Z}^d} \tilde{\eta}_t(z))$. The dual BD process is a random interface growing from a single seed; in this it resembles the classical first passage percolation (FPP) model, in which edges of the nearest-neighbor lattice $\mathbb{Z}^d$ carry independent identically distributed weights (representing passage times for each edge) and the agglomeration at time t consists of those sites (elements of $\mathbb{Z}^d$) accessible from the origin by paths through the lattice with total weight at most t.

Our proof of (2.10) uses ideas from the proof by Pemantle and Peres [13] of an analogous logarithmic lower bound for the variance of first passage times for FPP in $\mathbb{Z}^2$ with exponentially distributed edge weights. Using different methods, Newman and Piza [12] generalize this lower bound to FPP with a more general class of edge weights, and also show that the variance of first passage times in any direction of positive curvature diverges faster than $n^{(1/4)-\varepsilon}$ for any $\varepsilon > 0$. It would be interesting to try to extend the methods of [12] to obtain a lower bound for $w(t)$ with power law growth in the BD model.

The best *upper* bound on the growth rates of the variance in FPP seems to be the linearly growing bound of Kesten [11] (in the special case of an edge-weight distribution supported by two points, Benjamini *et al.* [2] provide a logarithmic improvement). It would be of interest to establish an upper bound for $w^2(t)$ with linear growth, but we have not done so. Further progress in estimating the growth rate of the variance for first passage times has proved elusive; by analogy, the same could well be true in the case of $w^2(t)$.

3 Continuum Ballistic Deposition

We consider a continuum ballistic deposition model, defined as follows. The substrate $\tilde{Q}$ is a Borel-measurable region of $\mathbb{R}^d$ (for example, a cube of side n) and $|\tilde{Q}|$ denotes the Lebesgue measure of $\tilde{Q}$. Particles are assumed to be $(d+1)$-dimensional Euclidean balls of possibly random independent identically distributed radii which are uniformly bounded by a finite constant $R_{\max}$ (in fact the results presented here could easily accommodate a more general class of random shapes, such as convex shapes of uniformly bounded diameter). Let F denote the common cumulative distribution function of the radii of incoming particles; assume that $F(0) = 0$ and $F(R_{\max}) = 1$.

Each incoming particle arrives perpendicularly to the substrate $\tilde{Q} \times \{0\}$ and sticks to the first previous particle it encounters, or to the substrate if it does not encounter any previous particle. In other words, its motion stops when it encounters a previous particle or the substrate, and remains stationary thereafter.

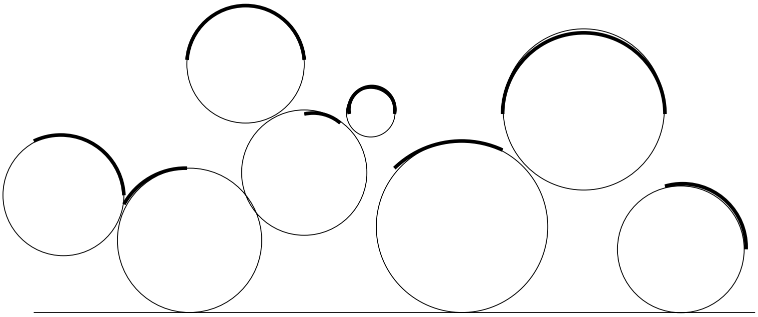

Fig. 2 The mean value of the function whose graph is given by the *bold arcs* is $\overline{H}_{t,n}$, and its variance is $W_{t,n}^2$. The *horizontal line* represents $\tilde{Q}_n \times \{0\}$

For simplicity we assume $\tilde{Q}$ is given by one of the sets $\tilde{Q}_n (n \geq 1)$ defined by

$$\tilde{Q}_n := Q_n \oplus [0,1)^d = \{x + y : x \in Q_n, y \in [0,1)^d\},$$

with the sequence of finite sets $Q_n \subset \mathbb{Z}^d$ assumed to satisfy the conditions (2.3)–(2.5).

Let $\mathcal{P}$ be a homogeneous Poisson point process of unit intensity in $\mathbb{R}^d \times [0, \infty)$, each point carrying a mark with distribution F. Denote by $\mathcal{P}_{t,n}$ the restriction of $\mathcal{P}$ to $\tilde{Q}_n \times [0, t]$, and denote by $\mathcal{P}_t$ the restriction of $\mathcal{P}$ to $\mathbb{R}^d \times [0, t]$.

Represent points in $\mathcal{P}$ by (X, T), where $X \in \mathbb{R}^d$ denotes the spatial location (center) of the incoming particle and T its time of arrival. Given n, t, the BD process driven by $\mathcal{P}_{t,n}$ is defined as follows. The spatial locations of incoming particles are given by the spatial locations of the points of $\mathcal{P}_{t,n}$, the order in which they arrive is determined by the time-coordinates of these points (i.e., they arrive in order of increasing time-coordinate), and their radii are given by the marks the points carry.

Let $A_{t,n}$ denote the agglomeration of particles for the BD process driven by $\mathcal{P}_{t,n}$, together with the substrate $\tilde{Q}_n \times \{0\}$ (a subset of $\mathbb{R}^{d+1}$). For each $x \in \tilde{Q}_n$ let $H_{t,n}(x)$ denote the height of the interface above x, i.e., let

$$H_{t,n}(x) := \sup\{h : (x, h) \in A_{t,n}\}. \tag{3.1}$$

The bold line in Fig. 2 represents the graph of the function $H_{t,n}(\cdot)$.

We define the average height $\overline{H}_{t,n}$ and width $V_{t,n}$ of the interface at time t as follows. We set $\overline{H}_{t,n}$ to be the mean of the function $H_{t,n}(x), x \in Q_n$, and $V_{t,n}$ to be the root-mean-square deviation of this function from $\overline{H}_{t,n}$. That is, we set

$$\overline{H}_{t,n} := |Q_n|^{-1} \int_{\tilde{Q}_n} H_{t,n}(x) dx; \tag{3.2}$$

$$V_{t,n} := \sqrt{|Q_n|^{-1} \int_{\tilde{Q}_n} (H_{t,n}(x) - \overline{H}_{t,n})^2 dx}. \tag{3.3}$$

These definitions of height and width are slightly different from those used in Penrose and Yukich [18] but no less natural.

The following result gives meaning to the height of the interface above an *infinite* substrate.

Proposition 3.1 *For all $t \geq 0$, $x \in \mathbb{R}^d$ the limit*

$$H_t(x) = \lim_{n\to\infty} H_{t,n}(x) \tag{3.4}$$

exists almost surely and is almost surely finite. Also, the distribution of $H_t(x)$ does not depend on x. Moreover, for all $k \in \mathbb{N}$ it is the case that $E[(H_t(\mathbf{0}))^k] = O(t^k)$ as $t \to \infty$.

Thus the infinite-substrate height function is given by $H_t(x)$, $x \in \mathbb{R}^d$. The following thermodynamic limits are variants of results in [18]. They are continuum analogues to Proposition 2.2. In particular, they show that the variance of the random variable $H_t(\mathbf{0})$ (height above a typical point of the infinite substrate) is the large-n limit of the width functionals $V_{t,n}^2$ (sample variance of the heights above a bounded substrate).

Proposition 3.2 *For all $t \in (0, \infty)$ and $p \in [1, \infty)$,*

$$\overline{H}_{t,n} \xrightarrow{L^p} E[H_t(\mathbf{0})] \quad \text{as } n \to \infty; \tag{3.5}$$

$$V_{t,n}^2 \xrightarrow{L^p} \mathrm{Var}[H_t(\mathbf{0})] \quad \text{as } n \to \infty. \tag{3.6}$$

Now we give a continuum analogue to Theorem 2.3

Theorem 3.3 *There is a constant $\rho_2 \in (0, \infty)$ such that $t^{-1}E[H_t(\mathbf{0})] \to \rho_2$ as $t \to \infty$. Moreover, for any $p \geq 1$,*

$$t^{-1}H_t(\mathbf{0}) \to \rho_2 \quad \text{in } L^p \text{ as } t \to \infty. \tag{3.7}$$

The next result is the continuum analogue to Theorem 2.4.

Theorem 3.4 *For any d, it is the case that*

$$\liminf_{t\to\infty} \mathrm{Var}[H_t(\mathbf{0})] > 0, \tag{3.8}$$

and if $d = 1$, then

$$\liminf(\mathrm{Var}[H_t(\mathbf{0})]/\log t) > 0. \tag{3.9}$$

Also of independent interest is the duality result given by Proposition 5.2 below, which is the continuum analogue of Theorem 2.5.

4 Proofs for Lattice BD

For $x \in \mathbb{Z}^d$, let $S_1(x), S_2(x), S_3(x), \dots$ denote the ordered arrival times of the Poisson process at site x. The proofs for lattice BD are based on a directed graph representation. For each $t > 0$, define a directed graph $\mathcal{G}_t$ and a directed graph $\tilde{\mathcal{G}}_t$, both with vertex set $\mathcal{V}_t$ defined by

$$\mathcal{V}_t := \{(z, S_i(z)) : z \in \mathbb{Z}^d, i \in \mathbb{N}, S_i(z) < t\} \cup \{(x, 0), (x, t); x \in \mathbb{Z}^d\}.$$

Informally, each Poisson arrival at site $x \in \mathbb{Z}^d$ before time t is represented by a point $(x, T) \in \mathbb{Z}^d \times [0, \infty)$. Two points (x, T) and (y, U) of $\mathcal{V}_t$ are joined by a directed edge in $\mathcal{G}_t$ from (x, T) to (y, U) if $T < U$ and $y \in \mathcal{N}_x$. They are joined by a directed edge in $\tilde{\mathcal{G}}_t$ from (x, T) to (y, U) if $T < U$ and $y \in \mathcal{N}_x^*$.

A *path* in $\mathcal{G}_t$ is a sequence π of vertices in $\mathcal{V}_t$ denoted $(x_0, T_0), \dots, (x_n, T_n)$ say, such that $T_0 = 0$ and $T_n = t$ and for $i = 1, 2, \dots, n$ there is an edge of $\mathcal{G}_t$ from (x_{i-1}, T_{i-1}) to (x_i, T_i). We say the path *starts* at $(x_0, 0)$ and *ends* at (x_n, t). The *length* of the path is n. The *height* of the path, denoted $h(\pi)$, is $\sum_{j=1}^n D(x_j - x_{j-1})$.

A path in $\tilde{\mathcal{G}}_t$ is defined similarly, except that now the requirement is that for each i there is an edge of $\tilde{\mathcal{G}}_t$ from (x_{i-1}, T_{i-1}) to (x_i, T_i), and the (dual) height $\hat{h}(\pi)$ of a path $\pi = ((x_i, T_i))_{i=0}^n$ in $\tilde{\mathcal{G}}_t$ is given by $\sum_{j=1}^n \hat{D}(x_j - x_{j-1})$, with $\hat{D}(\cdot)$ defined by (2.11).

The *skeleton* of a path π in $\mathcal{G}_t$ is its projection onto $\mathbb{Z}^d$, i.e., the sequence $(x_0, \dots, x_n)$. Given a sequence $(x_0, \dots, x_n) \in (\mathbb{Z}^d)^{n+1}$ with $x_i - x_{i-1} \in \mathcal{N}$ for $1 \le i \le n$, a *maximal path* with skeleton $(x_1, \dots, x_n)$ is a path $((x_0, T_0), \dots, (x_n, T_n))$ in the graph $\mathcal{G}_t$, with $T_0 = 0$ and $T_n = t$, and with the property that for $1 \le i \le n$, there are no Poisson arrivals at x_n in the time-interval (T_{i-1}, T_i).

We now express the next-arrival heights $\eta_t(z)$ for the BD process and $\hat{\eta}_t(z)$ for the dual BD process as defined in Sect. 2, in terms of the graphs $\mathcal{G}_t$ and $\tilde{\mathcal{G}}_t$.

Lemma 4.1 *Let $t > 0$ and $z \in \mathbb{Z}^d$. Suppose the maximum length of all paths in $\mathcal{V}_t$ ending at (z, t) is finite. Then $\eta_{t-}(z) := \lim_{s \uparrow t} \eta_s(z)$ is given by*

$$\eta_{t-}(z) = \sup\{h(\pi) : \pi \text{ a path in } \mathcal{G}_t \text{ ending at } (z, t)\}, \tag{4.1}$$

and $\hat{\eta}_{t-}(z) := \lim_{s \uparrow t} \hat{\eta}_s(z)$ is given by

$$\hat{\eta}_{t-}(z) = \sup\{\hat{h}(\pi) : \pi \text{ a path in } \tilde{\mathcal{G}}_t \text{ starting at } (\mathbf{0}, 0) \text{ and ending at } (z, t)\}, \tag{4.2}$$

with the convention that the supremum of the empty set is $-\infty$.

Proof If π is a finite path ending at (z, t), then we assert that $\eta_{t-}(z) \ge h(\pi)$. Indeed, if $\pi = (x_i, T_i)_{i=0}^n$ then by monotonicity of the processes ξ_t and η_t, for each i with $1 \le i < n$ we have

$$\xi_{T_i}(x_i) \ge \eta_{T_i-}(x_i) \ge \xi_{T_{i-1}}(x_{i-1}) + D(x_i - x_{i-1})$$

so that

$$\eta_{t-}(z) \ge \xi_{T_{n-1}}(x_{n-1}) + D(x_n - x_{n-1}) \ge \sum_{i=1}^n D(x_i - x_{i-1}) = h(\pi).$$

Conversely, there is at least one path of height at least $\eta_{t-}(z)$ that ends at (z, t). This is proved by induction on the maximum length of paths ending at (z, t). It is clearly true when this maximum path length is 1; suppose it is true when the maximum path length is in the range $\{1, 2, \dots, k\}$. Now suppose the maximum path length is $k + 1$. Then

$$\eta_{t-}(z) = \max\{\xi_{t-}(y) + D(z - y) : y \in \mathcal{N}_z^*\}$$

so that for some $y^* \in \mathcal{N}_z^*$ (i.e., with $z \in \mathcal{N}_{y^*}$) we obtain

$$\eta_{t-}(z) = \xi_t(y^*) + D(z - y^*),$$

and if L denotes the last Poisson arrival time before time t at site y^*, we have

$$\eta_{t-}(z) = \xi_L(y^*) + D(z - y^*) = \eta_{L-}(y^*) + D(z - y^*).$$

Each path in $\mathcal{G}_L$ ending at (y^*, L) has length at most k, since otherwise there would be a path through (y^*, L) to (z, t) of length greater than $k+1$. Hence by the inductive hypothesis, there is a path in $\mathcal{G}_L$ ending at (y^*, L), of height at least $\eta_{L-}(y^*)$, and hence by appending (z, t) to this sequence one obtains a path in $\mathcal{G}_t$, ending at (z, t), of height at least $\eta_{L-}(y^*) + D(z - y^*)$. This completes the induction and hence the proof (4.1). The proof of (4.2) is similar. $\square$

Proof of Proposition 2.1 Since $\xi_t(\mathbf{0}) \le \eta_t(\mathbf{0})$, it suffices to prove that $\eta_t(\mathbf{0})$ is almost surely finite and $E[\eta_t(\mathbf{0})^k] = O(t^k)$ as $t \to \infty$. Note that $\eta_t(\mathbf{0}) = \eta_{t-}(\mathbf{0})$ almost surely. Also, let $D_{\max} := \max_{z \in \mathbb{Z}^d} D(z)$ denote the maximum value of the displacement function. By our assumptions on this function, we have $1 \le D_{\max} < \infty$.

By Lemma 4.1, if $\eta_{t-}(\mathbf{0}) \ge m$ then there must be a path in $\mathcal{G}_t$ of height at least m, and hence of length at least $m/D_{\max}$, that ends at $(\mathbf{0}, t)$. Hence, if $\eta_{t-}(\mathbf{0}) \ge m$ then there is a path in $\mathcal{G}_t$ of length at least $m/D_{\max}$ that is maximal for its skeleton and ends at $(\mathbf{0}, t)$.

For any given sequence $(x_0, x_1, \dots, x_n)$ with $x_i - x_{i-1} \in \mathcal{N}$ for $i = 1, \dots, n$, the probability that there exists a maximal path in $\mathcal{G}_t$ with skeleton $(x_0, x_1, \dots, x_n)$ and with all arrival times less than t, equals than the probability that the sum of $n-1$ independent $\exp(1)$ variables is less than t, which is the same as the probability that $\mathrm{Po}(t) \ge n-1$, where $\mathrm{Po}(t)$ denotes a Poisson variable with mean t. Hence, for any $y \ge t\mathrm{e}^2 + 1$, by e.g. Lemma 1.2 of [14],

$$\begin{aligned} P[\eta_t(\mathbf{0}) \ge yD_{\max}] &= P[\eta_{t-}(\mathbf{0}) \ge yD_{\max}] \le |\mathcal{N}|^y P[\mathrm{Po}(t) \ge y-1] \\ &\le |\mathcal{N}|^y \exp(-((y-1)/2)\log((y-1)/t)) \end{aligned} \tag{4.3}$$

which tends to zero as $y \to \infty$. Hence $\eta_t(\mathbf{0})$ is almost surely finite. Also, for $k \in \mathbb{N}$,

$$E\left[\left(\frac{\eta_t(\mathbf{0})}{D_{\max}}\right)^k\right] = \int_0^\infty P\left[\frac{\eta_t(\mathbf{0})}{D_{\max}} \ge w^{1/k}\right] dw.$$

Set $c := (2|\mathcal{N}|)^2$, split the region of integration into $w \le (ct+1)^k$ and $w \ge (ct+1)^k$, and change variables to $y = w^{1/k}$ in the second integral to obtain the estimate

$$\begin{aligned} E\left[\left(\frac{\eta_t(\mathbf{0})}{D_{\max}}\right)^k\right] &\le (ct+1)^k + \int_{ct+1}^\infty P\left[\frac{\eta_t(\mathbf{0})}{D_{\max}} \ge y\right] ky^{k-1} dy \\ &\le (ct+1)^k + k\mathrm{e}^{1/2} \int_{ct+1}^\infty |\mathcal{N}|^y \exp(-(y/2)\log(c)) y^{k-1} dy \\ &\le (ct+1)^k + k\mathrm{e}^{1/2} \int_0^\infty y^{k-1} \exp(-y \log 2) dy \end{aligned}$$

and the last integral is finite, so that $t^{-k} E[\eta_t(\mathbf{0})^k]$ is bounded. $\square$

To prove Proposition 2.2 (and again in the next section), we shall use a slight generalization of a result of Penrose [16], which we now describe. Let $\mathcal{B}$ denote the collection of all non-empty finite subsets of $\mathbb{Z}^d$. Suppose $X = (X_x, x \in \mathbb{Z}^d)$ is a family of independent identically distributed random elements of a given measurable space. A *stationary $\mathcal{B}$-indexed* *summand* with respect to X is a family of random variables $(Y_z(Q) : Q \in \mathcal{B}, z \in Q)$ with the property that $(X_x, x \in Q)$ determines the value of $Y_z(Q)$ in a stationary (i.e. translation-invariant) manner (see section 3 of [16] for more details).

Lemma 4.2 *Let $p \geq 1$. Suppose $(Y_z(Q) : Q \in \mathcal{B}, z \in Q)$ is a stationary $\mathcal{B}$-indexed summand with respect to X, such that the p-th moments of $|Y_{\mathbf{0}}(Q)|$ are bounded uniformly over $Q \in \mathcal{B}$ with $\mathbf{0} \in Q$. Suppose there is a random variable Y such that $Y_{\mathbf{0}}(B_n) \xrightarrow{L^p} Y$ as $n \to \infty$, for any $\mathcal{B}$-valued sequence $(B_n)_{n \geq 1}$ with $\liminf_{n\to\infty}(B_n) = \mathbb{Z}^d$. If $(A_n)_{n\geq 1}$ is a $\mathcal{B}$-valued sequence with $|\partial A_n|/|A_n| \to 0$ as $n \to \infty$, then*

$$|A_n|^{-1} \sum_{z \in A_n} Y_z(A_n) \xrightarrow{L^p} E[Y] \quad \text{as } n \to \infty.$$

Proof The $p = 1$ version of this result is the first part of Theorem 3.1 of [16]. By changing L^1 estimates to L^p estimates throughout the proof of (3.3) of [16], we may deduce the same result with L^p convergence. The proof in [16] uses a multiparameter L^1 ergodic theorem quoted from [15], but this is also easy to extend to L^p convergence in the present setting, using the assumption of bounded pth moments. □

Proof of Proposition 2.2 Let $X = (X_x, x \in \mathbb{Z}^d)$ be a family of independent homogeneous Poisson processes of unit intensity. For $Q \in \mathcal{B}$, assume the evolution of $\xi_{t,Q}$ is governed by the Poisson processes $(X_x, x \in Q)$. Then $(\xi_{t,Q}(z), Q \in \mathcal{B}, z \in Q)$ is a stationary $\mathcal{B}$-indexed summand with respect to X. Also, the proof of Proposition 2.1 shows that for $p \in \mathbb{N}$

$$\sup\{E[\xi_{t,Q}(\mathbf{0})^{4p}] : Q \in \mathcal{B}, \mathbf{0} \in Q\} < \infty. \tag{4.4}$$

For any $\mathcal{B}$-valued sequence $(B_n, n \geq 1)$ with $\liminf(B_n) = \mathbb{Z}^d$, we have as $n \to \infty$ that $\xi_{t,B_n}(\mathbf{0}) \to \xi_t(\mathbf{0})$ almost surely (see the proof of Proposition 2.1, or Lemma 5.1 of [16]), and hence in L^p by (4.4). Hence, by Lemma 4.2 applied to the stationary $\mathcal{B}$-indexed summand $(\xi_{t,Q}(z), Q \in \mathcal{B}, z \in Q)$, we obtain (2.6).

To prove (2.7), first expand the sum of squares in (2.2) to obtain

$$W_{t,Q_n}^2 = |Q_n|^{-1}\left(\sum_{x \in Q_n} \xi_{t,Q_n}(x)^2\right) - \overline{\xi}_{t,Q_n}^2. \tag{4.5}$$

By (4.4), Lemma 4.2 is applicable to the stationary $\mathcal{B}$-indexed summand $(\xi_{t,Q}(x)^2, Q \in \mathcal{B}, x \in Q)$, and this shows that the first term in the right hand side of (4.5) converges in L^p to $E[\xi_t(\mathbf{0})^2]$.

Since (2.6), holds with convergence in L^{2p}, the second term in the right hand side of (4.5) converges in L^p to $(E[\xi_t(\mathbf{0})])^2$. Combining these limiting results in (4.5), we obtain (2.7). □

Proof of Theorem 2.5 The idea of the proof is a form of *time-reversal* of the graphical representation. Let $\psi_t : \mathbb{Z}^d \times [0, t] \to \mathbb{Z}^d \times [0, t]$ be defined by

$$\psi_t((z, s)) = (z, t - s).$$

Let $\hat{\mathcal{V}}_t := \psi_t(\mathcal{V}_t)$, and observe that the point set $\hat{\mathcal{V}}_t$ has the same distribution as $\mathcal{V}_t$ by the properties of Poisson point processes.

Let $\hat{\mathcal{G}}_t$ be defined in the same manner as $\tilde{\mathcal{G}}_t$ but on the vertex set $\hat{\mathcal{V}}_t$ instead of $\mathcal{V}_t$. Then each path in $\mathcal{G}_t$ ending at $(0, t)$ corresponds to a path with the same height starting at $(0, 0)$ in

$\hat{\mathcal{G}}_t$; the correspondence is obtained by reversing the sequence of vertices and then applying the mapping ψ_t to each vertex in the sequence.

Note that $\eta_t(\mathbf{0}) = \eta_{t-}(\mathbf{0})$ with probability 1. By (4.1), $\eta_{t-}(\mathbf{0})$ is the greatest height of all paths in $\mathcal{G}_t$ which end at $(\mathbf{0}, t)$. By the correspondence described above, this is precisely the same as the greatest height of all paths in the graph $\hat{\mathcal{G}}_t$ which start at $(\mathbf{0}, 0)$.

Since the point processes $\hat{\mathcal{V}}_t$ $\mathcal{V}_t$ have the same distribution, it follows that $\eta_t(\mathbf{0})$ has the same distribution as the greatest height of all paths in the graph $\tilde{\mathcal{G}}_t$ which start at $(\mathbf{0}, 0)$. Hence by (4.2), $\eta_t(\mathbf{0})$ has the same distribution as $\sup_{z \in \mathbb{Z}^d} \hat{\eta}_{t-}(z)$ and hence the same distribution as $\sup_{z \in \mathbb{Z}^d} \hat{\eta}_t(z)$.

Let L be the last arrival time at $\mathbf{0}$ before time t (or $L = 0$ if there are no arrivals at $\mathbf{0}$ before time t). Then $t - L$ has the distribution of $\min(T, t)$ where T is exponential with mean 1, and hence L has the distribution of $(t - T)^+$. Given L with $L > 0$, the distribution of $\eta_{L-}(\mathbf{0})$ is the same as that of $\sup_{z \in \mathbb{Z}^d} \hat{\eta}_L(z)$, by the same argument as above and the Markov property of the time-reversed Poisson process. Whenever $L > 0$ we have $\xi_t(\mathbf{0}) = \eta_{L-}(\mathbf{0})$, and if $L = 0$ then $\xi_t(\mathbf{0}) = 0$ almost surely. Combining these observations shows that $\xi_t(\mathbf{0})$ has the same distribution as $\sup_{z \in \mathbb{Z}^d} \hat{\eta}_{(t-T)^+}(z)$, as asserted. □

From now onwards, we shall assume the delayed dual BD (next-arrival) process $\tilde{\eta}_t(z)$ and also the associated last-arrival process $\tilde{\xi}_t(z)$ are defined in terms of the Poisson arrival times $(S_i(x), i \geq 1, x \in \mathbb{Z}^d)$, as follows. For $t < S_1(\mathbf{0})$ we put $\tilde{\eta}_t(z) = 0$ and $\tilde{\xi}_t(z) = -\infty$ for all $z \in \mathbb{Z}^d$. Then we put

$$\tilde{\xi}_{S_1}(z) := \begin{cases} 0 & \text{if } z = \mathbf{0}, \\ -\infty & \text{otherwise,} \end{cases}$$

and define $\tilde{\eta}_{S_1(\mathbf{0})}$ in terms of $\tilde{\xi}_{S_1(\mathbf{0})}$ in the usual manner as given at (2.1). Then we allow the evolution of $(\tilde{\xi}_t, \tilde{\eta}_t)_{t \geq S_1(\mathbf{0})}$ to follow the usual rules of the BD process with displacement function $\hat{D}$ given by (2.11), driven by the Poisson arrivals $\{S_i(z) : S_i(z) > S_1(\mathbf{0})\}$. Then $\tilde{\eta}_t(x), x \geq 0$ constructed in this manner clearly follows the desired evolution prescribed in Sect. 2, with $S_1(\mathbf{0})$ used as the initial 'kicking off time'.

For $t \geq 0, u \geq 0$, let D_t be the depth (i.e., the maximum next-arrival height) of the delayed dual BD process $\tilde{\eta}_t$ at time t, and let $T(u)$ be first passage time to depth u for the delayed dual BD process, i.e. let

$$D_t := \sup_{z \in \mathbb{Z}^d} \{\tilde{\eta}_t(z)\}; \qquad T(u) := \inf\{t : D_t \geq u\}.$$

Lemma 4.3 *There is a constant $\rho \in (0, \infty)$ such that*

$$\lim_{u \to \infty} \frac{T(u)}{u} = \rho, \quad a.s. \tag{4.6}$$

and

$$\lim_{t \to \infty} \frac{D_t}{t} = \rho^{-1}, \quad a.s. \tag{4.7}$$

Proof First we verify that $E[T(1)^2]$ is finite. Note that $\tilde{\eta}_t(\mathbf{0}) \geq N_t$, where here N_t denotes the number of arrivals at $\mathbf{0}$ up to time 1. Hence,

$$P[T(1) > t] \leq P[\tilde{\eta}_t(\mathbf{0}) < 1] \leq P[N_t < 2] = P[X > t]$$

where here X denotes an exponential random variable with unit mean, representing the first arrival time at $\mathbf{0}$. Hence, $E[T(1)^2] \le E[X^2] < \infty$.

Next, we assert that $T(u)$ is distributionally subconvolutive, i.e. for $u, v \ge 0$ we have

$$F_{T(u+v)} \ge F_{T(u)} * F_{T(v)}. \tag{4.8}$$

To see this, let $X(u) \in \mathbb{Z}^d$ be chosen (in an arbitrary way if there is more than one choice) so that $\tilde{\eta}_{T(u)}(X(u)) \ge u$ (by definition such $X(u)$ exists). Let $T(u) + T^*$ be the time of next Poisson arrival after $T(u)$ at site $X(u)$, and let $(\hat{\xi}_s^*, \hat{\eta}_s^*)_{s \ge 0}$ be a version of the BD process with displacement function $\hat{D}$, with initial profile

$$\hat{\xi}_0^*(x) = \begin{cases} 0 & \text{if } x = X(u), \\ -\infty & \text{otherwise} \end{cases}$$

and driven by Poisson arrival times $\{S_i^*(x)\}$ given for each $x \in \mathbb{Z}^d$ by

$$\{S_i^*(x)\} = \{S_i(x) - T(u) - T^* : S_i(x) > T(u) + T^*\},$$

where $\{S_i(x)\}$ are the arrival times driving the original dual BD process $(\hat{\xi}_t, \hat{\eta}_t)$.

Let $T^{**}(v)$ be the first time the process $(\hat{\eta}_s^*)_{s \ge 0}$ achieves a depth of at least v, i.e.

$$T^{**}(v) = \inf\Big\{t \ge 0 : \sup_{x \in \mathbb{Z}^d} (\hat{\eta}_t^*(x)) \ge v\Big\}.$$

Then $T^* + T^{**}(v)$ has the same distribution as $T(v)$, and is independent of $T(u)$. Also, since $\tilde{\xi}_{T(u)+T^*}(X(u)) \ge u$, the depth at time $T(u) + T^* + T^{**}(v)$ is at least $u + v$, i.e.

$$\sup_{x \in \mathbb{Z}^d} (\tilde{\eta}_{T(u)+T^*+T^{**}(v)}(x)) \ge u + v,$$

so that $T(u) + T^* + T^{**}(v) \ge T(u + v)$. Combining these facts gives us (4.8). Since the variables $(T(u), u \ge 0)$ are also monotonically increasing in u, we can apply the Kesten-Hammersley theorem ([21], page 20) to obtain (4.6), with $0 \le \rho < \infty$. Also,

$$\frac{D_t}{T(D_t)} \ge \frac{D_t}{t} \ge \frac{D_t}{D_t + 1} \times \frac{D_t + 1}{T(D_t + 1)} \tag{4.9}$$

and since $D_t \to \infty$ almost surely this with (4.6) yields (4.7) provided $\rho > 0$.

If $\rho = 0$ then by (4.6) and the second inequality of (4.9) we would have $D_t/t \to \infty$ almost surely, so that $E[D_t/t] \to \infty$, and so by Theorem 2.5, $E[\xi_t(\mathbf{0})/t] \to \infty$. This contradicts Proposition 2.1, and hence $\rho > 0$ as asserted. □

Proof of Theorem 2.3 Set $\rho_1 := \rho^{-1}$, with ρ as given in Lemma 4.3. By Theorem 2.5, D_t has the same distribution as $\xi_t(\mathbf{0})$, and so by Proposition 2.1, for any $p \ge 1$ the pth moment of (D_t/t) is bounded uniformly in t. By (4.7), (D_t/t) converges almost surely to ρ_1, and by the moment bound the almost sure convergence extends to convergence in pth moment for any $p \ge 1$. Since D_t has the same distribution as $\xi_t(\mathbf{0})$, this convergence in pth moment also holds for $\xi_t(\mathbf{0})$. □

Proof of Theorem 2.4 To prove (2.9) consider the event that (i) there are no arrivals in $\mathcal{N} \setminus \{\mathbf{0}\}$ between times $t - 1$ and t, and (ii) there is at least one arrival at $\mathbf{0}$ in the time-interval $(t - 1, t]$. This event has the same non-zero probability for all $t \ge 1$. Conditioned on this event, and on everything before time $t-1$, the conditional variance of $\xi_t(\mathbf{0})$ is the variance of the number of Poisson arrivals at $\mathbf{0}$ in the time-interval $(t-1,t]$, i.e. the variance of a Poisson variable with unit mean conditioned to take a value of at least 1. This variance is a strictly positive constant and (2.9) follows.

Now suppose $d=1$ and $\mathcal{N}$ is a lattice interval. To prove (2.10), we consider the delayed dual BD process $(\tilde{\xi}_t,\tilde{\eta}_t)_{t\geq 0}$. In this process, we denote by *accepted arrival* an arrival time $T=S_i(z)$ such that $\tilde{\xi}_T(z)>-\infty$. Enumerate the set of all accepted arrival times (for all sites) in increasing order as $\tau_1,\tau_2,\ldots$. Let N_t be the number of accepted arrivals up to time t, i.e.

$$N_t := \sup\{n : \tau_n \leq t\}.$$

Let I_t be the size of the interface at time t, i.e., the number of sites $z\in\mathbb{Z}$ with $\tilde{\eta}_t(z)>-\infty$. For $n\geq 1$, let $Y_n := I_{\tau_n}$ be the size of the interface after n accepted arrivals, and set $Y_0:=1$. Since $(N_t, t\geq 0)$ is a Poisson counting process with its 'clock' running at speed I_t, we have that

$$\lim_{t\to\infty}\frac{N_t}{\int_0^t I_u du}=1, \quad \text{a.s.} \tag{4.10}$$

Next, we assert that there is a constant $\gamma>0$ such that

$$\lim_{t\to\infty}(I_t/t)=\gamma, \quad \text{a.s.} \tag{4.11}$$

To see this, recall that we are assuming here that $\mathcal{N}$ is a lattice interval including $\mathbf{0}$ and at least one other element. It is not hard to see that I_t must also be a lattice interval, and that both the right and the left endpoint of I_t follow renewal reward processes, where in both cases the inter-arrival times of the underlying renewal process are exponentially distributed and the rewards are uniformly distributed over a lattice interval. The assertion (4.11) follows by the Strong Law of Large Numbers for a renewal reward process.

By (4.10) and (4.11) we obtain

$$\lim_{t\to\infty}\frac{N_t}{t^2}=\frac{\gamma}{2}, \quad \text{a.s.} \tag{4.12}$$

Let $M(u)$ be the number of accepted arrivals up to time $T(u)$. By (4.12) and (4.6), as $u\to\infty$, it is the case with probability 1 that

$$M(u)=N_{T(u)}\sim(\gamma/2)T(u)^2\sim(\gamma\rho^2/2)u^2. \tag{4.13}$$

Let $\mathcal{F}$ be the σ-algebra generated by the locations in $\mathbb{Z}$ of the sequence of accepted arrivals. Conditional on $\mathcal{F}$, the distribution of $T(u)$ is that of the sum of $M(u)$ independent exponentials with the jth exponential having mean Y_{j-1}^{-1}. Hence,

$$\sigma_u^2 := \mathrm{Var}[T(u)|\mathcal{F}]=\sum_{j=1}^{M(u)}Y_{j-1}^{-2}. \tag{4.14}$$

By definition, $N_{\tau_j}=j$, and hence by (4.12), $j/\tau_j^2\to\gamma/2$ so that $\tau_j\sim(2j/\gamma)^{1/2}$ as $j\to\infty$, almost surely. Since $Y_j=I_{\tau_j}$, by (4.11) we obtain as $j\to\infty$ that with probability 1,

$$Y_j\sim\gamma\tau_j\sim(2\gamma j)^{1/2}, \tag{4.15}$$

so that by (4.14) and (4.13), as $u \to \infty$ we have that

$$\sigma_u^2 \sim (2\gamma)^{-1} \log M(u) \sim \gamma^{-1} \log u, \quad \text{a.s.} \tag{4.16}$$

The Berry-Esseen theorem, e.g. as given in theorem 5.4 of Petrov [19] or as quoted in Chen and Shao [4], says that there is a constant C such that if $X_1, \dots X_k$ are independent random variables with mean zero and finite third moments and $W := \sum_{i=1}^k X_i$ has variance 1, then

$$\sup_{x\in\mathbb{R}} \{|P[W \le x] - \Phi(x)|\} \le C \sum_{i=1}^k E[|X_i|^3],$$

where Φ is the standard normal cumulative distribution function.

Let $\tau_0 := 0$, and for $i \ge 1$ let $e_i := \tau_i - \tau_{i-1}$. As mentioned earlier, conditional on $\mathcal{F}$ the e_i are independent exponentials with $E[e_i|\mathcal{F}] = Y_{i-1}^{-1}$. Define

$$\begin{aligned}\theta_u &:= \sum_{i=1}^{M(u)} E[|e_i - E[e_i|\mathcal{F}]|^3|\mathcal{F}] \\ &= (12\mathrm{e}^{-1} - 2) \sum_{i=1}^{M(u)} Y_{i-1}^{-3}, \end{aligned} \tag{4.17}$$

since if X is exponential with mean a then $E[|X - a|^3] = a^3(12\mathrm{e}^{-1} - 2)$.

Set $\mu_u := E[T(u)|\mathcal{F}]$. By the Berry-Esseen Theorem,

$$\sup_{x\in\mathbb{R}} \left| P\left[\frac{T(u) - \mu_u}{\sigma_u} \le x|\mathcal{F}\right] - \Phi(x)\right| \le \frac{C\theta_u}{\sigma_u^3}. \tag{4.18}$$

By (4.15), the sum $\sum_{i=1}^{\infty} Y_{i-1}^{-3}$ converges almost surely. Hence by (4.16) and (4.17), we can find u_0 such that for $u \ge u_0$ we have $P[A_u] < 0.01$ where

$$A_u := \left\{\frac{C\theta_u}{\sigma_u^3} > 0.01\right\} \cup \{\sigma_u < \sqrt{(\log u)/(2\gamma)}\}$$

so that $A_u \in \mathcal{F}$. Then for any $y \in \mathbb{R}$, using (4.18) we may deduce that

$$\begin{aligned} &P[T(u) \le y + 0.2(\gamma^{-1}\log u)^{1/2}|A_u^{\mathrm{c}}] - P[T(u) \le y|A_u^{\mathrm{c}}] \\ &\quad = P\left[\frac{T(u) - \mu_u}{\sigma_u} \le \frac{y - \mu_u}{\sigma_u} + \frac{0.2(\gamma^{-1}\log u)^{1/2}}{\sigma_u}|A_u^{\mathrm{c}}\right] \\ &\qquad - P\left[\frac{T(u) - \mu_u}{\sigma_u} \le \frac{y - \mu_u}{\sigma_u}|A_u^{\mathrm{c}}\right] \\ &\quad \le P\left[\frac{T(u) - \mu_u}{\sigma_u} \le \frac{y - \mu_u}{\sigma_u} + (0.2)\sqrt{2}|A_u^{\mathrm{c}}\right] - P\left[\frac{T(u) - \mu_u}{\sigma_u} \le \frac{y - \mu_u}{\sigma_u}|A_u^{\mathrm{c}}\right] \\ &\quad \le \sup_{x\in\mathbb{R}^d} \{\Phi(x + (0.2)\sqrt{2}) - \Phi(x)\} + 0.02 \le 0.22, \end{aligned}$$

and so for $u \ge u_0$ we have

$$P[T(u) \le y + 0.2(\gamma^{-1}\log u)^{1/2}] - P[T(u) \le y] \le 0.22 + P[A_u] < 1/4. \tag{4.19}$$

For $t > 0$, let $\nu(t)$ be a median of the distribution of D_t. Then

$$P[T(\nu(t)) > t] = P[D_t < \nu(t)] \leq 1/2 \tag{4.20}$$

so that $P[T(\nu(t)) \leq t] \geq 1/2$, and hence by (4.19), for t large enough

$$P[T(\nu(t)) \leq t - 0.2(\gamma^{-1}\log\nu(t))^{1/2}] \geq 1/4.$$

By (4.7), $(D_t/t) \to \rho^{-1}$ in probability, so that $(\nu(t)/t) \to \rho^{-1}$ as $t \to \infty$. Hence,

$$\log\nu(t) \sim \log t \quad \text{as } t \to \infty. \tag{4.21}$$

If $T(\nu(t)) \leq t - 0.2(\gamma^{-1}\log\nu(t))^{1/2}$, then there is a site $z^* \in \mathbb{Z}$ with

$$\tilde{\eta}_{t-0.2(\gamma^{-1}\log\nu(t))^{1/2}}(z^*) \geq \nu(t).$$

If also at least $0.1(\gamma^{-1}\log t)^{1/2} + 1$ particles arrive at site z^* between times $t - 0.2(\gamma^{-1}\log\nu(t))^{1/2}$ and t, then $D_t \geq \nu(t) + 0.1(\gamma^{-1}\log t)^{1/2}$. By (4.21), the conditional probability of the second of these events, given the first, tends to 1 as $t \to \infty$, so that for large t,

$$P[D_t \geq \nu(t) + 0.1(\gamma^{-1}\log t)^{1/2}] > 1/8. \tag{4.22}$$

Moreover, by definition $P[D_t \leq \nu(t)] \geq 1/2$. Combining this with (4.22) yields (2.10). □

5 Proofs for Continuum BD

In this section, for $x \in \mathbb{R}^d$ and $r > 0$ we write $B_r(x)$ for the closed Euclidean ball of radius r centered at x.

We introduce a *dual continuum BD* process, denoted $\hat{H}_t(x)$, defined in an identical manner to the original continuum BD process except that now the initial profile $\hat{H}_0(x)$, $x \in \mathbb{R}^d$ is given by

$$\hat{H}_0(x) = \begin{cases} 0 & \text{if } x = \mathbf{0}, \\ -\infty & \text{otherwise.} \end{cases}$$

In other words, the interface grows from an initial seed consisting of a single point at $(\mathbf{0}, 0)$. Incoming particles which miss the agglomeration have no effect.

We adapt the graphical representation to the continuum. Given a locally finite marked point set $\mathcal{X} \subset \mathbb{R}^d \times [0, \infty)$, we say a sequence of points $(X_i, T_i)_{i=1}^k$ in $\mathcal{X}$ forms a *path in* $\mathcal{X}$ if $T_1 < T_2 < \cdots < T_k$, and for $1 \leq i < k$, the d-dimensional ball centered at X_i (of radius given by the mark of (X_i, T_i)) overlaps the ball centered at X_{i+1} (of radius given by the mark of (X_{i+1}, T_{i+1})).

Given such a path, we refer to the sequence $(X_1, \ldots, X_k)$ as the *skeleton* of the path. Given also $x \in \mathbb{R}^d$, we say the path *starts near* x if x lies in the d-dimensional ball centered at X_1 (of radius given by the corresponding mark) and we say the path *ends near* x if x lies in the d-dimensional ball centered at X_n. We say the *height* at x of the path is the height of the interface above x for the BD agglomeration onto an initially flat surface determined by the finite sequence of incoming $(d+1)$-dimensional balls centered at $X_1, \ldots, X_k$ (with radii given by the corresponding marks).

If the path starts near $\mathbf{0}$, we say the *dual height* of the path at x is the height of the interface above x for the BD agglomeration determined by the finite sequence of incoming

$(d+1)$-dimensional balls centered at $X_1, \dots, X_k$ (with radii given by the corresponding marks), starting with the initial profile being given by the function $\hat{H}_0$ (i.e. by a single point at $(\mathbf{0}, 0)$ rather than an initially flat surface).

Lemma 5.1 *Let* $t \geq 0$, $x \in \mathbb{R}^d$. *Then with probability* 1:

(i) *There are almost surely only finitely many paths in* $\mathcal{P}_t$ *which end near* x;
(ii) *the value of* $H_t(x)$ *is the maximum height at* x *of all such paths* (*or is zero if no such path exists*);
(iii) $H_{t,n}(x)$ *is the maximum height at* x *of all paths in* $\mathcal{P}_t$ *which end near* x *for which the skeleton is contained in* $\tilde{Q}_n$ (*or is zero if no such path exists*), *and*
(iv) $\hat{H}_t(x)$ *is the maximum* dual *height at* x *of all paths which start near* $\mathbf{0}$ *and end near* x (*or is* $-\infty$ *if no such path exists*).
(v) *There exists an almost surely finite random* $N = N(t)$ *such that* $H_t(x) = H_{t,n}(x)$ *for all* $x \in [0, 1)^d$ *and all* $n \geq N$.

Proof Part (i) follows from e.g. Corollary 3.1 of [15]. We remark here that this result in fact shows that with probability 1 it is the case that for all $x \in [0, 1)^d$, there are only finitely many paths ending near x.

In the continuum BD process, there exists a sequence of particles in the agglomeration, each particle touching the next one in the sequence, leading from the substrate to a particle arriving at time at most t and with $(x, H_t(x))$ on its surface in the agglomeration; considering only the particles in this sequence, we have a path which ends near x and has height $H_t(x)$ at x. Hence the height $H_t(x)$ is at most the maximum height at x of all such paths.

On the other hand, inserting extra points into a given $(d+1)$-dimensional marked point process cannot decrease the height over x of the interface of the corresponding BD agglomeration, so for any path in $\mathcal{P}_t$ ending near x, $H_t(x)$ is at least the height at x of the path. This demonstrates part (ii).

The argument for part (iii) is the same as for part (ii), except that now one ignores particles with spatial location outside $\tilde{Q}_n$.

The argument for part (iv) is the same as for part (ii).

Part (v) follows from parts (ii) and (iii), together with the remark at the end of the proof of part (i). $\square$

Proof of Proposition 3.1 Let $x \in \mathbb{R}^d$. By (2.4) and part (i) of Lemma 5.1, there exists an almost surely finite random n_0 such that for $n \geq n_0$, every path in $\mathcal{P}_t$ which ends near x has its skeleton contained in $\tilde{Q}_n$. Then by parts (ii) and (iii) of Lemma 5.1, we have $H_t(x) = H_{t,n}(x)$ for $n \geq n_0$. This demonstrates the first part of Proposition 3.1.

Since $\mathcal{P}_t$ is distributionally invariant under spatial translations (i.e., translations of $\mathbb{R}^d \times \mathbb{R}$ leaving the time-coordinate unchanged), the distribution of $H_t(x) := \lim_{n\to\infty} H_{t,n}(x)$ does not depend on x. This demonstrates the second part of Proposition 3.1.

We prove the last part only in the case where $R_{\max} \leq 1/2$; the more general case can then be deduced by some simple scaling arguments which we omit.

We couple our continuum BD process to a certain NNN lattice BD model, as defined in Section 2. Partition $\mathbb{R}^d$ into half-open unit cubes, and for $x \in \mathbb{R}^d$ let $Q(x)$ be the cube in the partition that contains x. Let $(\xi_t, \eta_t)_{t\geq 0}$ be the coupled NNN lattice BD model, in which the arrival times at site $z \in \mathbb{Z}^d$ are given by the time-coordinates of the points of $\mathcal{P}$ in $Q(z) \times (0, \infty)$. By the assumption that $2R_{\max} \leq 1$, it is not hard to see that $H_t(x) \leq \eta_t(z(x))$. We can then use Proposition 2.1 to deduce that $E[(H_t(\mathbf{0}))^k] = O(t^k)$. $\square$

Proof of Proposition 3.2 For $z \in \mathbb{Z}^d$ let $C_z := \{z + y : y \in [0,1)^d\}$. Let $X = (X_z, z \in \mathbb{Z}^d)$ be the family of independent homogeneous Poisson point processes of unit intensity in $[0,1)^d \times [0,\infty)$, obtained by taking X_z to be the image of the restriction of $\mathcal{P}$ to $C_z \times [0,\infty)$ under the translation $(X, T) \mapsto (X - z, T)$. For $Q \in \mathcal{B}$, let $\tilde{Q} := \{x + y : x \in Q, y \in [0,1)^d\}$, and for $z \in Q$, set

$$Y_{t,z}(Q) := \int_{C_z} H_{t,Q}(u)du \tag{5.1}$$

where $H_{t,Q}(u)$ denotes the height of the interface above u at time t if we ignore the points of $\mathcal{P}$ lying outside $\tilde{Q} \times [0,\infty)$ (so for example, $H_{t,Q_n}(u) = H_{t,n}(u)$ as defined at (3.1)).

As in the proof of Proposition 2.2, we aim to apply Lemma 4.2; we can do so because $(Y_{t,z}(Q), z \in Q, Q \in \mathcal{B})$ is a stationary $\mathcal{B}$-indexed summand with respect to X. Also, for $p \in \mathbb{N}$ the proof of Proposition 3.1 shows that

$$\sup\{E[H_{t,Q}(u)^{4p}] : Q \in \mathcal{B}, u \in \mathbb{R}^d\} < \infty, \tag{5.2}$$

so by Fubini's theorem and Hölder's inequality

$$\sup\{E[Y_{t,\mathbf{0}}(Q)^{4p}] : Q \in \mathcal{B}, \mathbf{0} \in Q\} < \infty. \tag{5.3}$$

For any $\mathcal{B}$-valued sequence $(B_n, n \geq 1)$ with $\liminf(B_n) = \mathbb{Z}^d$, by Lemma 5.1 (v) we have almost surely as $n \to \infty$ that

$$Y_{t,\mathbf{0}}(B_n) \to Y_{t,\mathbf{0}} := \int_{C_\mathbf{0}} H_t(u)du, \tag{5.4}$$

and by (5.3) this convergence also holds in L^p. Hence, Lemma 4.2 is applicable to the stationary $\mathcal{B}$-indexed summand $(Y_{t,x}(Q), Q \in \mathcal{B}, x \in Q)$, yielding

$$\overline{H}_{t,n} = |Q_n|^{-1} \sum_{z \in Q_n} Y_{t,z}(Q_n) \xrightarrow{L^p} EY_{t,\mathbf{0}}. \tag{5.5}$$

By Proposition 3.1 the distribution of $H_t(x)$ does not depend on x, so by (5.4) and Fubini's theorem, $EY_{t,\mathbf{0}} = E[H_t(\mathbf{0})]$ so (5.5) yields (3.5).

We now prove (3.6). By expanding out the square in (3.3), we obtain the identity

$$V_{t,n}^2 = \left(|Q_n|^{-1} \int_{\tilde{Q}_n} H_{t,n}^2(x)dx \right) - \overline{H}_{t,n}^2 = \left(|Q_n|^{-1} \sum_{z \in Q_n} Y_{t,z}^{(2)}(Q_n) \right) - \overline{H}_{t,n}^2, \tag{5.6}$$

where we set

$$Y_{t,z}^{(2)}(Q) := \int_{C_z} (H_{t,Q}(u))^2 du.$$

For any $\mathcal{B}$-valued sequence $(B_n, n \geq 1)$ with $\liminf(B_n) = \mathbb{Z}^d$, by Lemma 5.1 (v) we have almost surely as $n \to \infty$ that

$$Y_{t,\mathbf{0}}^{(2)}(B_n) \to Y_{t,\mathbf{0}}^{(2)} := \int_{C_\mathbf{0}} (H_t(u))^2 du. \tag{5.7}$$

By (5.2), Fubini's theorem and Hölder's inequality, the $2p$th moment of $Y_{t,\mathbf{0}}^{(2)}(Q)$ is bounded, uniformly over $Q \in \mathcal{B}$ with $\mathbf{0} \in Q$, so the convergence (5.7) also holds in L^p. Hence, Lemma 4.2 is applicable to the stationary $\mathcal{B}$-indexed summand $(Y_{t,z}^{(2)}(Q), Q \in \mathcal{B}, z \in Q)$, and this shows that

$$|Q_n|^{-1} \sum_{z \in Q_n} Y_{t,z}^{(2)}(Q_n) \xrightarrow{L^p} E[Y_{t,\mathbf{0}}^{(2)}] = E[(H_t(\mathbf{0}))^2]$$

where the equality follows from the definition (5.7) of $Y_{t,\mathbf{0}}^{(2)}$, Fubini's theorem and the fact that the distribution of $H_t(u)$ does not depend on u (Proposition 3.1).

Since (3.5) holds with convergence in L^{2p}, the second term in the right hand side of (5.6) converges in L^p to $(E[H_t(\mathbf{0})])^2$. Combining these limiting results in (5.6), we obtain (3.6). □

For $t \geq 0$ and $u \geq 0$, let $\tilde{D}_t$ denote the depth (i.e. maximum height) of the dual continuum BD model at time t, and let $\tilde{T}(u)$ be the first passage time to depth u of the dual continuum BD model. i.e. let

$$\tilde{D}_t := \sup_{x \in \mathbb{R}^d} (\hat{H}_t(x)); \quad \tilde{T}(u) := \inf\{t \geq 0 : D_t \geq u\}.$$

Proposition 5.2 *The distribution of $H_t(\mathbf{0})$ is the same as that of $\tilde{D}_t$.*

Proof Fix $t > 0$, and consider the time-reversed space-time Poisson process with time-coordinates transformed by the mapping $(X, T) \mapsto (X, t - T)$. Under this mapping, any path $((X_1, T_1), \ldots, (X_k, T_k))$ corresponds (by reversing the order of points) to a path in the transformed Poisson process, namely $((X_k, t - T_k), \ldots, (X_1, t - T_1))$, the so-called *time-reversed path*. If the original path ends near $\mathbf{0}$, the corresponding time-reversed path starts near $\mathbf{0}$, and the height at $\mathbf{0}$ of the original path equals the dual height over X_1 of the time-reversed path, which is the maximal dual height of the time-reversed path.

By Lemma 5.1 (ii), $H_t(\mathbf{0}) \geq u$ if and only if there is a path in $\mathcal{P}_t$ which ends near $\mathbf{0}$ with height at $\mathbf{0}$ of at least u by time t, in which case the corresponding time-reversed path has maximal height at least u. Hence, $H_t(\mathbf{0})$ is the maximal dual height of time-reversed paths in $\mathcal{P}_t$ starting near $\mathbf{0}$. Hence by Lemma 5.1 (iv), $H_t(\mathbf{0})$ is the maximal depth at time t for the continuum ballistic deposition process driven by the transformed Poisson process, using initial profile $\hat{H}_0$. Since the distributions of the original and transformed Poisson processes are identical, $H_t(\mathbf{0})$ therefore has the same distribution as maximal depth in the BD process generated by the original Poisson process with initial profile $\hat{H}_0$. In other words, it has the same distribution as $\tilde{D}_t$. □

The next result is a continuum analogue to Lemma 4.3.

Lemma 5.3 *There is a constant $\rho_3 \in (0, \infty)$ such that*

$$\lim_{u \to \infty} \frac{\tilde{T}(u)}{u} = \rho_3, \quad a.s. \tag{5.8}$$

and

$$\lim_{t \to \infty} \frac{\tilde{D}_t}{t} = \rho_3^{-1}, \quad a.s. \tag{5.9}$$

Proof First we show that $\tilde{T}(1)$ has finite second moment. Choose $\varepsilon_1 > 0$ such that $F(\varepsilon_1) \leq 1/2$, and $\varepsilon_2 > 0$ such that a ball of radius ε_1 in $\mathbb{R}^{d+1}$ contains a rectilinear cube of side ε_2, with the same center. Let $N_1(t)$ be the number of Poisson arrivals in $\mathcal{P}_t$ having spatial coordinate with ℓ_∞ norm at most $\varepsilon_2/2$ and having mark at least ε_1. Then $N_1(t)$ is Poisson with parameter at least $\varepsilon_2 t/2$, and $\hat{H}_t(\mathbf{0}) \geq \varepsilon_2 N_1(t)$, so that

$$P[\tilde{T}(1) > t] \leq P[\hat{H}_t(\mathbf{0}) < 1] \leq P[\mathrm{Po}(\varepsilon_2 t/2) < 1/\varepsilon_2] = P\bigg[\sum_{1\leq i\leq \lceil 1/\varepsilon_2\rceil} e'_i > t\bigg]$$

where $e'_1, e'_2, e'_3, \ldots$ are independent exponential random variables with mean $2/\varepsilon_2$, representing the inter-arrival times of a Poisson process of rate $\varepsilon_2/2$, and $\lceil x\rceil$ denotes the smallest integer not less than x. Hence,

$$E[\tilde{T}(1)^2] \leq E\bigg[\bigg(\sum_{1\leq i\leq \lceil 1/\varepsilon_2\rceil} e'_i\bigg)^2\bigg] < \infty.$$

We assert that $\tilde{T}(u)$ is distributionally subconvolutive, i.e. for $u, v \geq 0$ we have

$$F_{\tilde{T}(u+v)} \geq F_{\tilde{T}(u)} * F_{\tilde{T}(v)}. \tag{5.10}$$

To see this, let $X(u) \in \mathbb{R}^d$ be chosen (in an arbitrary way if there is more than one choice) so that $\hat{H}_{\tilde{T}(u)}(X(u)) \geq u$ (by definition such $X(u)$ exists). Let $(\hat{H}^*_s(x), x \in \mathbb{R}^d)_{s\geq 0}$ be a version of the BD process with initial profile

$$\hat{H}^*_0(x) = \begin{cases} 0 & \text{if } x = X(u), \\ -\infty & \text{otherwise} \end{cases}$$

and driven by the Poisson process $\tau_{-\tilde{T}(u)}(\mathcal{P}) \cap (\mathbb{R}^d \times (0,\infty))$, where τ_{-t} denotes the shift operator on $\mathbb{R}^d \times \mathbb{R}$ mapping each point $(x,u) \in \mathbb{R}^d \times \mathbb{R}$ to $(x, u-t)$.

Let $\tilde{T}^*(v)$ be the first time the process $(\hat{H}^*_s)_{s\geq 0}$ achieves a depth at least v, i.e.

$$\tilde{T}^*(v) = \inf\Big\{t \geq 0 : \sup_{x\in\mathbb{R}^d} (\hat{H}^*_t(x)) \geq v\Big\}.$$

Then $\tilde{T}^*(v)$ has the same distribution as $\tilde{T}(v)$, and is independent of $\tilde{T}(u)$. Also the depth at time $\tilde{T}(u)+\tilde{T}^*(v)$ is at least $u+v$, i.e. $\tilde{D}_{\tilde{T}(u)+\tilde{T}^*(v)} \geq u+v$, so that $\tilde{T}(u)+\tilde{T}^*(v) \geq \tilde{T}(u+v)$. Combining these facts gives us (5.10).

Since the variables $(\tilde{T}(u), u \geq 0)$ satisfy (5.10) and are also monotonically increasing in u, we can apply the Kesten-Hammersley theorem ([21], p. 20) to obtain the desired conclusion (5.8), with $0 \leq \rho_3 < \infty$.

The arguments to show that (5.9) holds and $\rho_3 > 0$ are just the same as the corresponding arguments in the proof of Lemma 4.3. □

Proof of Theorem 3.3 The proof is entirely analogous to that of Theorem 2.3, now using Propositions 3.1 and 5.2 along with Lemma 5.3. □

Proof of Theorem 3.4 First we prove (3.8). Let $\mathcal{F}_t$ be the σ-field generated by all arrivals up to time $t-1$. Then $H_{t-1}(\mathbf{0})$ is $\mathcal{F}_t$-measurable. It is clear that there is a constant $\varepsilon > 0$ such that

$$P[H_t(\mathbf{0}) = H_{t-1}(\mathbf{0}) | \mathcal{F}_t] \geq \varepsilon \quad \text{a.s.}$$

and

$$P[H_t(\mathbf{0}) \geq H_{t-1}(\mathbf{0}) + 1|\mathcal{F}_t] \geq \varepsilon \quad \text{a.s.}$$

and combining these two estimates shows that the conditional variance $\mathrm{Var}[H_t|\mathcal{F}_t]$ is bounded away from zero. From this we may deduce (3.8).

The proof of (3.9) is similar to that of (2.10). Now take I_t to be the Lebesgue measure of the interface, i.e., of the set of sites $x \in \mathbb{R}^d$ with $\hat{H}_t(x) > -\infty$, and take $\mathcal{F}$ to be the σ-algebra generated by the sequence of locations and marks of accepted arrivals in the dual continuum BD process.

Then it is again the case that conditional on $\mathcal{F}$, the distribution of $T(u)$ is the sum of $M(u)$ independent exponentials $e_1, \dots, e_{M(u)}$; this is because, given $\mathcal{F}$, the e_j are independent exponentials. Indeed, given the positions of the first j accepted arrivals, the distribution of e_j is exponential with mean Y_j^{-1}; extra information about the location of this arrival and subsequent arrivals does not affect its distribution.

Using this information, the proof of (3.9) follows that of (2.10) closely, and we omit further details. □

References

1. Atar, R., Athreya, S., Kang, M.: Ballistic deposition on a planar strip. Electron. Commun. Probab. **6**, 31–38 (2001)
2. Benjamini, I., Kalai, G., Schramm, O.: First passage percolation has sublinear distance variance. Ann. Probab. **31**, 1970–1978 (2003)
3. Barabási, A.-L., Stanley, H.E.: Fractal Concepts in Surface Growth. Cambridge University Press, Cambridge (1995)
4. Chen, L., Shao, Q.-M.: Normal approximation under local dependence. Ann. Probab. **34**, 1985–2028 (2004)
5. Cumberland, D.J., Crawford, R.J.: The Packing of Particles. Handbook of Powder Technology, vol. 6. Elsevier, Amsterdam (1987)
6. Family, F.: Scaling of rough surfaces: effects of surface diffusion. J. Phys. A **19**, L441–L446 (1986)
7. Family, F., Vicsek, T.: Scaling of the active zone in the Eden process on percolation networks and the ballistic deposition model. J. Phys. A **18**, L75–L81 (1985)
8. Gravner, J., Tracy, C.A., Widom, H.: Limit theorems for height fluctuations in a class of discrete space and time growth models. J. Stat. Phys. **102**, 1085–1132 (2001)
9. Gravner, J., Tracy, C.A., Widom, H.: A growth model in a random environment. Ann. Probab. **30**, 1340–1368 (2002)
10. Kardar, M., Parisi, G., Zhang, Y.: Dynamic scaling of growing interfaces. Phys. Rev. Lett. **56**(9), 889–892 (1986),
11. Kesten, H.: On the speed of convergence in first-passage percolation. Ann. Appl. Probab. **3**, 296–338 (1993)
12. Newman, C.M., Piza, M.S.T.: Divergence of shape fluctuations in two dimensions. Ann. Probab. **23**, 977–1005 (1995)
13. Pemantle, R., Peres, Y.: Planar first-passage percolation times are not tight. In: Grimmett, G. (ed.) Probability and Phase Transition, pp. 261–264. Kluwer, Dordrecht (1994)
14. Penrose, M.: Random Geometric Graphs. Oxford University Press, Oxford (2003)
15. Penrose, M.D.: Random parking, sequential adsorption, and the jamming limit. Commun. Math. Phys. **218**, 153–176 (2001)
16. Penrose, M.D.: Limit theorems for monotonic particle systems and sequential deposition. Stoch. Process. Appl. **98**, 175–197 (2002)
17. Penrose, M.D., Yukich, J.E.: Limit theory for random sequential packing and deposition. Ann. Appl. Probab. **12**, 272–301 (2001)
18. Penrose, M.D., Yukich, J.E.: Mathematics of random growing interfaces. J. Phys. A: Math. Gen. **34**, 6239–6247 (2001)
19. Petrov, V.V.: Limit Theorems of Probability Theory: Sequences of Independent Random Variables. Oxford University Press, Oxford (1995)

20. Seppäläinen, T.: Strong law of large numbers for the interface in ballistic deposition. Ann. Inst. H. Poincaré Probab. Stat. **36**, 691–735 (2000)
21. Smythe, R.T., Wierman, J.: First-Passage Percolation on the Square Lattice. Lecture Notes in Mathematics, vol. 671, Springer, Berlin (1978)
22. Vicsek, T.: Fractal Growth Phenomena. World Scientific, Singapore (1989)
23. Vold, M.: A numerical approach to the problem of sediment volume. J. Colloid Sci. **14**, 168–174 (1959)
24. Zabolitzky, J.G., Stauffer, D.: Simulation of large Eden clusters. Phys. Rev. A **34**(2), 1523–1530 (1986)